Factoring of $(a^2 + b^2)$

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1 Factoring

I had a student approach me stating that he saw online that factoring $(a^2 + b^2)$ had no solutions. So, I showed the student how we could factor this polynomial into two binomials. So let us consider:

$$(a^2 + b^2)$$

Since we know the first element is a^2 we automatically know it will look something like:

$$(a+?) * (a+?)$$

Now we need to think about what two numbers can we use so that they multiply to b^2 and that when we add up the products of a and the two numbers that they cancel each other out. What we have is:

$$(a + bi) * (a - bi)$$

So how do we know if this will work? If we multiply our binomials using the distributive property and we get back to our original polynomial then we have successfully factored out the two binomials that make up the polynomial. If we consider our binomials $(a + bi) * (a - bi)$ and use the distributive property to multiply the first element of the first binomial and the first element of the second binomial $(\mathbf{a} + bi) * (\mathbf{a} - bi)$ we get $a * a = a^2$.

If we now use the distributive property to multiply the first element of the first binomial and the second element of the second binomial $(\mathbf{a} + bi) * (a - \mathbf{bi})$ we get $a * (-bi) = -abi$.

If we continue and use the distributive property to multiply the second element of the first binomial and the first element of the second binomial $(a+\mathbf{bi})*(\mathbf{a}-bi)$ we get $(bi) * a = +abi$.

If we take our last step using the distributive property to multiply the second element of the first binomial and the second element of the second binomial

$(a + \mathbf{bi}) * (a - \mathbf{bi})$ we get $(+bi) * (-bi) = -(b^2 * i^2)$ But, if we remember our imaginary numbers we know that $i^2 = -1$ so this now can be simplified to $-(b^2 * i^2) = -(b^2 * -1) = -(-b^2) = +b^2$.

If we consider all of the products we derived using the distributive property we get:

$$a^2 - abi + abi + b^2$$

Now if we simplify we get: $a^2 + b^2$

1.1 Example

Now we have shown that indeed we are able to factor any such polynomial in the form

$$(a^2 + b^2)$$

to get the two binomial factors. We can now use another example of this to show again how this would work by considering:

$$(x^2 + 49)$$

Since we know the first element is x^2 we automatically know it will look something like:

$$(x+?) * (x+?)$$

Now we need to think about what two numbers can we use so that they multiply to 49 and that when we add up the products of x and the two numbers that they cancel each other out. What we have is:

$$(x + 7i) * (a - 7i)$$

So how do we know if this will work? Well, if we multiply our binomials using the distributive property and we will back to our original we have successfully factored out the two binomials that make up the polynomial. If we consider our binomials $(x + 7i) * (x - 7i)$ and use the distributive property to multiply the first element of the first binomial and the first element of the second binomial $(\mathbf{x} + 7i) * (\mathbf{x} - 7i)$ we get $x * x = x^2$.

If we now use the distributive property to multiply the first element of the first binomial and the second element of the second binomial $(\mathbf{x} + 7i) * (x - \mathbf{7i})$ we get $x * (-7i) = -x7i$.

If we continue and use the distributive property to multiply the second element of the first binomial and the first element of the second binomial $(x + \mathbf{7i}) * (\mathbf{x} - 7i)$ we get $(7i) * x = +x7i$.

If we take our last step using the distributive property to multiply the second element of the first binomial and the second element of the second binomial

$(x + \mathbf{7i}) * (x - \mathbf{7i})$ we get $(+7i) * (-7i) = -(7^2 * i^2)$ But, if we remember our imaginary numbers we know that $i^2 = -1$ so this now can be simplified to $-(7^2 * i^2) = -(49 * -1) = -(-49) = +49$.

If we consider all of the products we derived using the distributive property we get:

$$x^2 - x7i + x7i + 49$$

Now if we simplify we get: $x^2 + 49$

2 Try Some Examples

To try out your skills please consider using what you learned above to factor the following polynomials. The answers will be in the next section so you can check your answers.

2.1 Problem One

Factor the following polynomial into the two binomial components:

$$x^2 + 64$$

2.2 Problem Two

Factor the following polynomial into the two binomial components:

$$81 + x^2$$

3 Answers

3.1 Answer to problem one

$$(x + 8i) * (a - 8i)$$

3.2 Answer to problem two

$$(9 + xi) * (9 - xi)$$