Proof That The Group $Z_3 \rtimes Z_4$ Is A Non-Abelian Group

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We will prove that $Z_3 \rtimes Z_4$ is a non abelian group. We first note that $Z_3 \rtimes Z_4$ is a group of order 12 with $Z_3 \rtimes Z_4 = \langle x, y | x^4 = y^3 = 1, x^{-1}yx = y^{-1} \rangle$. The group $Z_3 \rtimes Z_4$ is a group with the invarient factors $3 \cdot 2^2$.

The first prime 3 is of the order p^1 and by applying Sylow's theorem so that $Z_3 \rtimes Z_4 = 3^{\alpha} \cdot (m)$ with $\alpha = 1$ & $m = (2^2)$ we then can see that

$$n_3 | (2^2) \quad n_3 \equiv 1 (\mathbf{mod}\, 3) \Rightarrow n_3 = 1 \text{ or } n_3 = 4$$

The second prime 2 is of the order p^2 and by applying Sylow's theorem so that $Z_3 \rtimes Z_4 = 2^{\alpha} \cdot (m)$ with $\alpha = 1$ & $m = (3)$ we then can see that

$$n_2 | (3) \quad n_2 \equiv 1 (\mathbf{mod}\, 2) \Rightarrow n_2 = 1 \text{ or } n_2 = 3$$

If $n_3 \neq 1$, then there are $n_3 (3-1) \cdot 4 = 8$ non-identity elements. With the identity element we have 9 elements in this subgroup. If $n_2 \neq 1$ then there are $n_2 (2-1) \cdot 2 = 2$ non-identity elements. With the identity element we have 3 elements in this subgroup. Since (9+3)=12 and the order of the group $Z_3 \rtimes Z_4$ is 12. This implies $Syl_3 (Z_3 \rtimes Z_4) \ntrianglelefteq (Z_3 \rtimes Z_4)$ & $Syl_2 (Z_3 \rtimes Z_4) \ntrianglelefteq (Z_3 \rtimes Z_4)$, and we can *not* show that by proposition 13 (Dummit, D. S., & Foote, R. M., pg 93, (2004)) , that $(|Syl_3(Z_3 x Z_4) \cap Syl_2(Z_3 x Z_4)|) = 1$.

Since $\dfrac{G}{C_G(P_5)} \neq 1$ implies that all the elements do not commute with $Z(G)$, this implies $Z_3 \rtimes Z_4$ is not cyclic and therefore $Z_3 \rtimes Z_4$ is non-abelian.

References

Dummit, D. S., & Foote, R. M. (2004). Abstract algebra. Hoboken, NJ: Wiley.
The Integers 1 to 10000, in groups of 100. (n.d.). Retrieved from http://www.positiveintegers.org/IntegerGroups/1-10000