When Is The Computation Of The Inverse Matrix Useful?

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Here we will discover that the computation of the inverse matrix is useful tool to understand the underlying system being modeled in the matrix. At a very fundamental level one reason to compute the inverse matrix is to test if it meets the technical criteria of a group. The technical definition of a group is:

"A group is an ordered pair $(G, \star)$ where G is a set and $\star$ is a binary operation on G satisfying the following axioms:

(i) $(a \star b) \star c = a \star (b \star c)$, for all $a, b, c \in G$, i.e., $\star$ is associative.
(ii) there exists an element e in G, called an identity of G, such that for all $a \in G$ we have $a \star e = e \star a = a$.
(iii) for each $a \in G$ there is an element a^{-1} of G, called an inverse of a, such that $a \star a^{-1} = a^{-1} \star a = e$." (Dummitt & Foote,2004, pg16-17)

By computing the inverse matrix we would be testing (iii) in the above definition. What this means here is that for each element a that exists in group G there is an inverse so that when you take the operation given $a \star a^{-1}$ you get the identity e. Here the element a is the matrix A, the inverse element a^{-1} is the inverse matrix A^{-1} and the identity element e is the identity matrix I.

Building on this one would want to show that the system meets the criteria of a field. The technical definition of a field is:

"A field is a set F together with two commutative binary operations + and $*$ on F such that $(F, +)$ is an abelian group (Call its identity 0) and $(F - \{0\})$ is also an abelian group, and the following distributive law holds:
$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$, for all $a, b, c \in F$."(Dummit & Foote, 2004, pg 34)

Computing the inverse matrix helps us in showing that the system meets this criteria as well. Specifically, by confirming that the determinant of the matrix does not equal zero and that there exists an inverse matrix then we show that this is a group under matrix multiplication.

So why bother testing to see if this system is a field? It turns out that fields will have certain properties. If we are dealing with a systems represented by polynomials and it is a field then we know that that it will at most the same number of roots as the degree of the polynomial. For example $2x^4+5x^3+4x^2+x+5$=0 is a polynomial whose degree is 4 would have at most four roots.(Dummit 2004)

In addition, if we have a linear system with an inverse matrix then we know that when we solve the system it will have a unique solution for x. (Kolman & Hill 2005) This is especially useful for non-homogeneous systems. (A homogeneous system is one that is equal to zero.) One application of this would be in the manufacturing world. We can consider manufacturing system which there were three steps in the manufacturing process for each of the final three pieces of a product. We can determine our production capacity and at what productions levels to generate output at each step for these final three pieces so there are exactly the number of final piece 1, final piece 2 and final piece 3 needed for final assembly without having extra inventory for either of the three final pieces.
If for pieces our production capacity at each piece was modeled as piece one = $3x_1+5x_2+4x_3=2$, piece two $x_1+5x_2+x_3=3$ and piece three $3x_1+4x_2+3x_3=4$ we would have the following system in matrix form and we would have the following unique solution.

$restart$:
$with(linalg)$:
$$A := \begin{bmatrix} 3 & 5 & 4 & 2 \\ 1 & 5 & 2 & 3 \\ 3 & 4 & 3 & 4 \end{bmatrix}:$$
$rref(A)$;

$$\begin{bmatrix} 1 & 0 & 0 & \frac{23}{8} \\ 0 & 1 & 0 & \frac{11}{8} \\ 0 & 0 & 1 & -\frac{27}{8} \end{bmatrix} \quad \textbf{(1)}$$

Now if one wanted to expand production, one might want to look for bottleneck areas in production and improve capacity at those points. Then we could go back with our revised model and get a new unique solution with a higher production capacity level. While this example was a simple manufacturing example the models can get quite large. Another area where this technique is often useful is with linear economic models.

The applications also extend to "linear first-order systems"(Greenberg pg 219) of ordinary differential equations. An example of a "real world" application of this would be in disaster recovery situations. Let's assume a water reservoir of a oceanside community has been contaminated by a storm surge of sea water. The water reservoir consists of two lakes in which water can be pumped from lake M to lake N through one pumping system and from lake N to lake M on another pumping system. Both pumping systems pump at the exact same rate. Since disaster recovery equipment is scarce a desalination device treatment system is brought to the location of lake M and where there is a pump pumping water into the device from the reservoir and another pump pumping desalinated water back into the reservoir. Another device is set up at the location of lake N that chlorinates the water to kill bacteria. Again, there is a pump pumping water into the device and another pump pumping the chlorinated water back into the reservoir. If we know the initial conditions of the water contamination, we can use this system of two ordinary differential equations and we can solve to see how long the system with both water reservoir pumps pumping to and from lake M and N will meet desired indicators and the locals can start using their water again. If the two differential equations below represented the two changes to the water over time we would have:

$x' = 3y + 2e^t$

$y' = 2x - e^t$

with initial conditions of $y(0) = 5,\ x(0) = 1$

$restart;\ Odeee := \{diff(x(t), t) = 3 \cdot y(t) + 2 \cdot e^t,\ diff(y(t), t) = 2 \cdot x(t) - e^t,\ y(0) = 5,\ x(0) = 1\};$

$$\left\{ \frac{d}{dt} x(t) = 3\,y(t) + 2\,e^t,\ \frac{d}{dt} y(t) = 2\,x(t) - e^t,\ x(0) = 1,\ y(0) = 5 \right\} \quad \textbf{(2)}$$

$dsolve(Odeee, \{x(t), y(t)\});$

$$\left\{ x(t) = e^{\sqrt{6}\,t}\left(\frac{2}{5} + \frac{7}{5}\sqrt{6}\right) + e^{-\sqrt{6}\,t}\left(\frac{2}{5} - \frac{7}{5}\sqrt{6}\right) + \frac{1}{5}e^t,\ y(t) = \frac{1}{3}\sqrt{6}\,e^{\sqrt{6}\,t}\left(\frac{2}{5} + \frac{7}{5}\sqrt{6}\right) - \frac{1}{3}\sqrt{6}\,e^{-\sqrt{6}\,t}\left(\frac{2}{5} - \frac{7}{5}\sqrt{6}\right) - \frac{3}{5}e^t \right\} \quad \textbf{(3)}$$

The answer here given by maple is the unique solution to this problem based on the given initial conditions with t representing time. While the example here involved two reservoirs mixing, these applications extend to many engineering problems that use ordinary differential equations including electrical circuitry. (Greenberg 2012)

If a matrix has an inverse then we know we can compute the solution using the $A = LU$ method. Doing so allows us to lower our big O time from $O(n^3)$ to $O(n^2)$ processing time. (Faires & Burden 1998)

If a matrix has an inverse and the inverse is equal to the transpose of the matrix then we know the matrix is orthogonal. In addition, we will know the eigenvectors of the orthogonal matrix will be orthogonal as well. Also, if we normalize the eigenvectors and create a new matrix from the eigenvectors, it too will be orthogonal. (Kolman & Hill 2005)

Determining that the inverse matrix exists is important to knowing various properties of the underlying system of equations and thus allowing us to solve array of problems that exist in nature and our modern society.

References

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