Proof that if $\sigma \in S_n$ is an element of odd order then $\sigma = A_n\sigma$.

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We assume $\sigma \in S_n$ is an element of odd order such that $(n \geq 3)$ and will prove $\sigma = A_n\sigma$ by using the First Isomorphism Theorem (Dummit, D. S., & Foote, R. M.,pg 97 (2004)) so that $\frac{S_n}{A_n} \cong \epsilon(S_n) = \{\pm 1\}$ & $\left|A_n\right| = \frac{1}{2}\left|S_n\right| = \frac{1}{2}\left(n!\right)$.

First let us look at S_3. The permuations are:
(123)(132)(12)(13)(23). If we consider whether each permutation is even or odd we get:
(1)(1)(-1)(-1)(-1)
Therefore we have shown that $\sigma \in S_3 = -1$
Also $\left|A_3\right| = \frac{1}{2}\left|S_3\right| = \frac{1}{2}\left(3!\right) = \frac{6}{2} = 3$ which is odd.

We assume $\frac{S_{2k+1}}{A_{2k+1}} \cong \epsilon(S_{2k+1}) = \{-1\}$ & $\left|A_{2k+1}\right| = \frac{1}{2}\left|S_{2k+1}\right| = \frac{1}{2}\left(\left(2k+1\right)!\right) = odd$
If add an even number $(2k)$ we get:

$$\frac{S_{2k+1+2k}}{A_{2k+1+2k}} \cong \epsilon(S_{2k+1+2k}) = \{-1\} \ \& \ \left|A_{2k+1+2k}\right| = \frac{1}{2}\left|S_{2k+1+2k}\right| = \frac{1}{2}\left(\left(2k+1+2k\right)!\right) = odd$$

$$\frac{S_{4k+1}}{A_{4k+1}} \cong \epsilon(S_{4k+1}) = \{-1\} \ \& \ \left|A_{4k+1}\right| = \frac{1}{2}\left|S_{4k+1}\right| = \frac{1}{2}\left(\left(4k+1\right)!\right) = odd$$

Therefore when n is odd $\sigma \in S_n$ is an element of odd order and $\sigma = A_n\sigma$ is odd (-1).

Now we can consider S_4
Odd Permutations = -1
(1234)
(1243)
(1423)
(1342)
(1324)
(1432)
(12)
(13)
(14)
(23)
(24)

(34)

There are 12 odd perutations (-1)

Even Permutations = 1
(123)
(124)
(132)
(134)
(142)
(143)
(234)
(243)
(12)(34)
(13)(24)
(14)(23)

There are 11 even permutations (1)

Since (-1)(-1)=(1) & 12/2,=6 we have shown that S_4 is even $\sigma = (1)$.

Also $\left|A_4\right| = \frac{1}{2}\left|S_4\right| = \frac{1}{2}\left(4!\right) = \frac{24}{2} = 12$ which is odd.

We assume $\frac{S_{2k}}{A_{2k}} \cong \epsilon(S_{2k}) = \{1\}$ & $\left|A_{2k}\right| = \frac{1}{2}\left|S_{2k}\right| = \frac{1}{2}\left(\left(2k\right)!\right) = even$
If add an even number $(2k)$ we get
$\frac{S_{2k+2k}}{A_{2k+2k}} \cong \epsilon(S_{2k+2k}) = \{1\}$ & $\left|A_{2k+2k}\right| = \frac{1}{2}\left|S_{2k+2k}\right| = \frac{1}{2}\left(\left(2k+2k\right)!\right) = even$

$\frac{S_{4k}}{A_{4k}} \cong \epsilon(S_{4k}) = \{1\}$ & $\left|A_{4k}\right| = \frac{1}{2}\left|S_{4k}\right| = \frac{1}{2}\left(\left(4k\right)!\right) = even$

Therefore when n is even $\sigma \in S_n$ is an element of even order and $\sigma = A_n\sigma$ is odd (1).

Therefore, since we have shown that when n is odd $\sigma = -1$ and when n is even then $\sigma = 1$, we conclude that $\sigma = A_n\sigma$.

References:
Dummit, D. S., & Foote, R. M. (2004). Abstract algebra. Hoboken, NJ: Wiley.