Why Numerical Methods for Integration are powerful.

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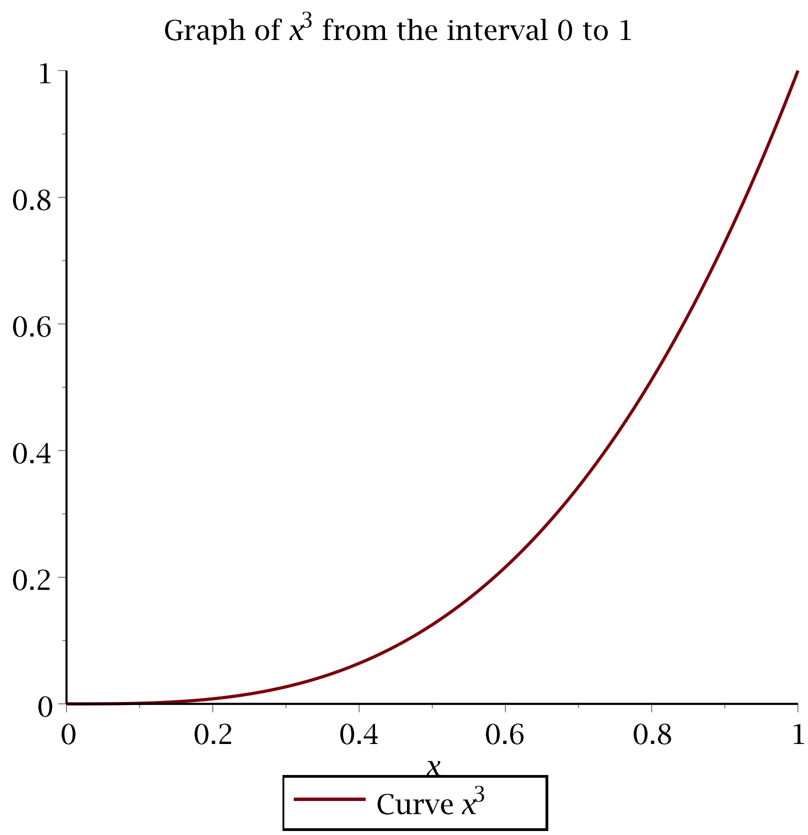
The general problem of finding an area has been around for some time. Techniques to estimate have been used for centuries. Integration calculus deals with the area problem in which one would want to find the area under the curve of a continuous function. The foundations were published in "Issac Barrow's Lectiones geometricae" (Anton, H., Bivens, I., & Davis, S. pg 321, (2002)) The technique behind integration is to take the area of rectangles as they approach infinity. To illustrate this idea using areas of rectangles to approximate the area under the curve we will estimate the area under a curve with three and then six rectangles. We can consider an example with the curve of x^3:



If we consider the basic area of a rectangle $(area = base \cdot height)$ that our base width is $\frac{(b-a)}{n} = \frac{(1-0)}{3} = \frac{1}{3}$ and we consider our height is $f(x^3) = \sum_{i=1}^{n} f\left(\left(b - \frac{i}{3}\right)^3\right)$ then we have

$$\left(\left(\frac{1}{3}\right)\cdot\left(1-\frac{1}{3}\right)^3\right) + \left(\left(\frac{1}{3}\right)\cdot\left(1-\frac{2}{3}\right)^3\right) + \left(\left(\frac{1}{3}\right)\left(1-\frac{3}{3}\right)^3\right) = \frac{4}{27} \approx 0.14815$$

By using larger values of n the rectangles get smaller and smaller and more closely approximate the actual area. Here we will show what happens if we double our number of rectangles to 6 rectangles:

$$\left(\left(\left(\frac{1}{6}\right)\cdot\left(1-\frac{1}{6}\right)^3\right) + \left(\left(\frac{1}{6}\right)\cdot\left(1-\frac{2}{6}\right)^3\right) + \left(\left(\frac{1}{6}\right)\left(1-\frac{3}{6}\right)^3\right) + \left(\left(\frac{1}{6}\right)\cdot\left(1-\frac{4}{6}\right)^3\right) + \left(\left(\frac{1}{6}\right)\cdot\left(1-\frac{5}{6}\right)^3\right) + \left(\left(\frac{1}{6}\right)\cdot\left(1-\frac{6}{6}\right)^3\right)\right) = \frac{17}{108} \approx 0.15741$$

The actual value of $\int_0^1 x^3 \, dx$ is (0.25). While this solution gives us a rough idea of our area we can see that the accuracy is far from ideal for practical use. To improve on the accuracy level we can consider other more effective techniques. One such technique is the trapezoidal rule:

"$\int_a^b f(x)\,dx = (b-a)\cdot\left(\frac{(f(a)+f(b))}{2}\right) - \frac{f^{(2)}(\xi)}{12}\cdot(b-a)^3$"(Faires, J. D., & Burden, R. L., pg133, (1998))

Now if we use the trapezoidal rule in our above example we have:

$$\int_0^1 f(x^3)\,dx = (1-0)\cdot\left(\frac{(f(0)+f(1))}{2}\right) = (1)\cdot\left(\frac{(0+1)}{2}\right) = (0.5)$$

Again, we can see that this technique is good to get a rough idea but is not ideal. The reason the result was not better than the last technique is that we considered the two points and obtained a trapezoid. The intervals were not broken into smaller pieces. To improve on this technique we can consider the composite trapezoidal rule:

"$\int_a^b f(x)\,dx = \left(\frac{h}{2}\right)\cdot\left[f(a) + f(b) + 2\cdot\sum_{j=1}^{n-1} f(x_j)\right] - \left(\frac{(b-a)h^2}{12}\cdot f^{(2)}(\gamma)\right)$"(Faires, J. D., & Burden, R. L., pg144, (1998)) Here we change our step size to get a closer approximation. Now if we take our step size to be $h = 0.1414213562$ & $n = 8$ we then get a result of ≈ 0.3843106778. So by going from one trapezoid to approximate the area to 8 smaller trapezoids we have significantly improved our results.

If we keep going and take an even smaller step size of $h = 0.04472135955$ & $n = 23$ we then get a result of ≈ 0.2783966797. So by going from 8 small trapezoids to 23 smaller trapezoids we have again significantly improved our results.

If we continue one more time and reduce our stepsize further so $h = 0.01414213562$ & $n = 71$ we now get a result of ≈ 0.2540800676. Which is quite close to our actual result. So we see here with this technique that if we increase the number of trapezoids we take, the result quickly approaches the actual value of the integral.

While the composite trapezoidal technique is powerful, it's limitation lies in the fact that it is of order 1. A more powerful technique is Romberg Integration which expands on these ideas. If we consider the general stepsize equation " $M - N(h) = K_1 h + K_2 h^2 + K_3 h^3 \ldots$"(Faires, J. D., & Burden, R. L., pg151, (1998)), then we end up with an estimated calculation that is in line with polynomials that has higher orders. The technique used for the Romberg integration takes initial calculations using the trapezoidal rule with:

" $R_{k,1} = \frac{1}{2} \cdot \left[R_{k-1,1} + h_{k-1} \cdot \sum_{i=1}^{2^{k-2}} f\left(a + (2\,i - 1) \cdot h_k\right) \right]$" (Faires, J. D., & Burden, R. L., pg154, (1998))

Afterwards, averaging is used to get the higher orders needed $\left(K_2 h^2 + K_2 h^3 \ldots etc\right)$ so that:

" $R_{k,j} = R_{k,j-1} + \frac{\left(R_{k,j-1} - R_{k-1,j-1}\right)}{4^{j-1} - 1}$"(Faires, J. D., & Burden, R. L., pg155, (1998))

If we calculate our above example using this technique then we get the following results:

$R_{1,1} = \frac{1}{2}$

$R_{2,1} = \frac{5}{16}$

$R_{2,2} = \frac{1}{4}$

By calculating only $R_{2,2}$ we were able to obtain our exact result. While in this case we obtained an exact result we may only closely approximate an equation at times. By using the higher orders we have significantly improved our results with fewer calculations.

References

Faires, J. D., & Burden, R. L. (1998). Numerical methods. Pacific Grove, CA: Brooks/Cole Pub. Co.

Anton, H., Bivens, I., & Davis, S. (2002). Calculus. New York [u.a.: Wiley.