Comparing Interpolation Methods

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There are numerous interpolation methods. Here we will look at a data set of four data points and use three interpolation methods and compare to see which had the least actual error and relative error. We will then assume further information and explore a fourth method.
Let's assume you are given the following data points:

x	$f(x)$
2	1.127625965
2.5	1.081072372
3	1.056071868
4	1.031413100

and we want to find a good value for when $x = 3.5$. One interpolation method would be to use Lagrange interpolating polynomials. The formula for a third degree polynomial is as follows:

$$P_3(x) = \left(\frac{((x-x_1)\cdot(x-x_2)\cdot(x-x_3))}{((x_0-x_1)\cdot(x_0-x_2)\cdot(x_0-x_3))}\right)\cdot f(x_0) + \left(\frac{((x-x_0)\cdot(x-x_2)\cdot(x-x_3))}{((x_1-x_0)\cdot(x_1-x_2)\cdot(x_1-x_3))}\right)\cdot f(x_1) + \left(\frac{((x-x_0)\cdot(x-x_1)\cdot(x-x_3))}{((x_2-x_0)\cdot(x_2-x_1)\cdot(x_2-x_3))}\right)\cdot f(x_2) + \left(\frac{((x-x_0)\cdot(x-x_1)\cdot(x-x_3))}{((x_3-x_0)\cdot(x_3-x_1)\cdot(x_3-x_2))}\right)\cdot f(x_3)$$

Now we can place in the appropriate values to our formula go get our polynomial:

$$P_3(x) = -1.127625965\,(x-2.5)\,(x-3)\,(x-4) + 2.882859659\,(x-2)\,(x-3)\,(x-4) - 2.112143736\,(x-2)\,(x-2.5)\,(x-4) + 0.3438043667\,(x-2)\,(x-2.5)\,(x-3)$$

And if we simplify we get:

$$P_3(x) = -0.013105675\,x^3 + 0.14139875\,x^2 - 0.52953999\,x + 1.72595635$$

We can now use this to estimate $x = 3.5$ as follows:

$$P_3(3.5) = -0.013105675\cdot(3.5^3) + 0.14139875\cdot(3.5^2) - 0.52953999\cdot(3.5) + 1.72595635$$

$$P_3(3.5) = -0.013105675\cdot(42.875) + 0.14139875\cdot(12.25) - 0.52953999\cdot(3.5) + 1.72595635$$

$$P_3(3.5) = (-0.561905815) + (1.732134688) - (1.853389965) + 1.72595635$$

$$P_3(3.5) = (1.042795258)$$

We could use Newton's interpolatory divided-difference formula to generate an interpolating polynomial in which we would get:

$$PN_3(x) = f[x_0] + f[x_0, x_1]\cdot(x-x_0) + f[x_0, x_1, x_2]\cdot(x-x_0)\cdot(x-x_1) + f[x_0, x_1, x_2, x_3]\cdot(x$$

$-x_0)\cdot(x-x_1)\cdot(x-x_2)$

We need to calculate $f[x_0, x_1]$:

$$f[x_0, x_1] = \frac{(f[x_1]-f[x_0])}{x_1-x_0} = \frac{(1.081072372-1.127625965)}{2.5-2} = \frac{(-.046553593)}{.5}$$
$$= (0.093107186)$$

Now we need to calculate $f[x_0, x_1, x_2]$

$$f[x_0, x_1, x_2] = \frac{(f[x_2, x_1]-f[x_1, x_0])}{x_2-x_0}$$

We already know that $f[x_0, x_1] = (0.093107186)$.

$$f[x_2, x_1] = \frac{(f[x_2]-f[x_1])}{x_2-x_1} = \frac{(1.056071868-1.081072372)}{3-2.5} = \frac{(-0.024353692)}{.5} = (-0.048707384)$$

Now we can calculate $f[x_0, x_1, x_2]$

$$f[x_0, x_1, x_2] = \frac{((-0.048707384)-(0.093107186))}{3-2} = \frac{(0.044399802)}{1}$$
$$= (0.044399802)$$

Now we need to calculate $f[x_0, x_1, x_2, x_3]$

$$f[x_0, x_1, x_2, x_3] = \frac{(f[x_1, x_2, x_3]-f[x_0, x_1, x_2])}{x_3-x_0}$$

We already know $f[x_0, x_1, x_2] = (0.044399802)$

$$f[x_1, x_2, x_3] = \frac{(f[x_2, x_3]-f[x_1, x_2])}{x_3-x_1}$$

We already know $f[x_1, x_2] = (-0.048707384)$

$$f[x_2, x_3] = \frac{(f[x_3]-f[x_2])}{x_3-x_2} = \frac{(1.03113100-1.056071868)}{4-3} = \frac{(-0.024658768)}{.1} = (-0.024658768)$$

Now we can calculate $f[x_1, x_2, x_3]$

$$f[x_1, x_2, x_3] = \frac{((-0.024658768)-(-0.048707384))}{4-2.5} = \frac{(0.024048616)}{1.5}$$
$$= (0.01603241)$$

Now we can calculate $f[x_0, x_1, x_2, x_3]$

$$f[x_0, x_1, x_2, x_3] = \frac{((0.01603241) - (0.044399802))}{4 - 2} = \frac{(-0.028367392)}{2} = (-0.014183696)$$

From here, we can place in the appropriate values to our formula go get our polynomial:

$$PN_3(x) = 1.127625965 + (0.093107186) \cdot (x - 2) + (0.044399802) \cdot (x - 2) \cdot (x - 2.5) + (-0.014183696) \cdot (x - 2) \cdot (x - 2.5) \cdot (x - 3)$$

We can now use this to estimate $x = 3.5$ as follows:

$$PN_3(3.5) = 1.127625965 + (0.093107186) \cdot (3.5 - 2) + (0.044399802) \cdot (3.5 - 2) \cdot (3.5 - 2.5) + (-0.014183696) \cdot (3.5 - 2) \cdot (3.5 - 2.5) \cdot (3.5 - 3)$$

$$PN_3(3.5) = 1.127625965 + (0.093107186) \cdot (1.5) + (0.044399802) \cdot (1.5) \cdot (1) + (-0.014183696) \cdot (1.5) \cdot (1) \cdot (0.5)$$

$$PN_3(3.5) = 1.127625965 + (0.139660779) + (0.066599703) + (-0.010637772)$$

$$PN_3(3.5) = 1.323248675$$

Another method we can use is the Hermite interpolation to generate a 3rd degree interpolating polynomial. The formula for this is:

$$H_3(x) = f[x_0] + f'[x_0] \cdot (x - x_0) \cdot (x - 2) + f[x_0, x_1, x_2] \cdot (x - x_0)^2 + f[x_0, x_1, x_2, x_3] \cdot (x - 2)^2 \cdot (x - x_1)$$

We will assume you only have the four data points and do not have the first derivative. We can closely approximate $f'[x_0]$ with $f[x_0, x_1]$ and our formula just becomes:

$$H_3(x) = f[x_0] + f[x_0, x_1] \cdot (x - x_0) + f[x_0, x_1, x_2] \cdot (x - x_0)^2 + f[x_0, x_1, x_2, x_3] \cdot (x - x_0)^2 \cdot (x - x_1)$$

Now we can place in the appropriate values to our formula go get our polynomial:

$$H_3(x) = 1.127625965 + (0.093107186) \cdot (x - 2) + (0.044399802) \cdot (x - 2)^2 + (-0.014183696) \cdot (x - 2)^2 \cdot (x - 2.5)$$

We can now use this to estimate $x = 3.5$ as follows:

$$H_3(3.5) = 1.127625965 + (0.093107186) \cdot (3.5 - 2) + (0.044399802) \cdot (3.5 - 2)^2 + (-0.014183696) \cdot (3.5 - 2)^2 \cdot (3.5 - 2.5)$$

$$H_3(3.5) = 1.127625965 + (0.093107186) \cdot (1.5) + (0.044399802) \cdot (2.25) + (-0.014183696) \cdot (2.25) \cdot (1)$$

$$H_3(3.5) = 1.127625965 + (0.139660779) + (0.099899554) + (-0.031913316)$$

$$H_3(3.5) = 1.127625965 + (0.139660779) + (0.099899554) + (-0.031913316)$$

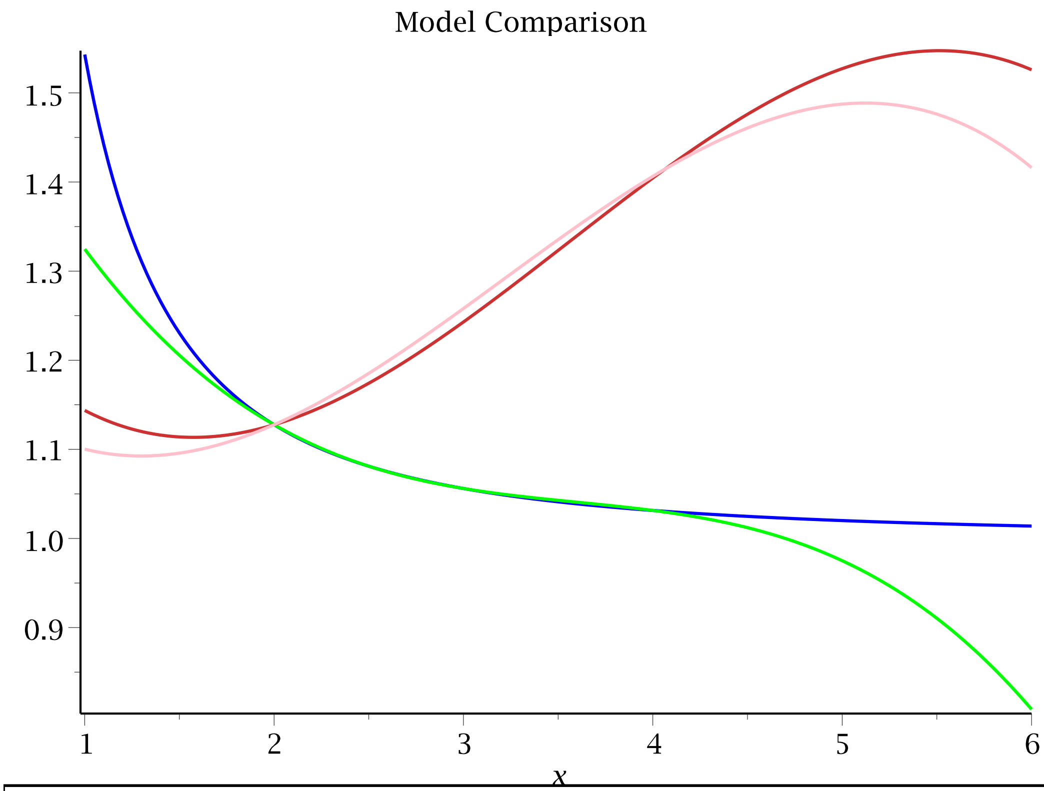
$$H_3(3.5) = 1.335272982$$

It turns out the underlying model is

$\cosh\left(\frac{1}{x}\right)$ and the actual value for x at 3.5 is 1.041094745. So what does this say about our interpolation models? On the surface we can look at the actual error based on this model and that we only had four data points to work with. We can compare the error for our model using the below data:

Model	Absolute Error	Relative Error
$P_3(x)$	0.0017000513	0.001630725
$PN_3(x)$	0.28215393	0.271016573
$H_3(x)$	0.294178237	0.282566249

In *this case* the Lagrange Interpolation yielded significantly lower levels than the other two models. We can look at how the actual model and the three interpolation models between the values of x of between 1 and 6 to get a feel for how these models behave.



$$f(x) = \left(\cosh\left(\frac{1}{x}\right)\right)$$

$$P_3(x) = -0.013105675\, x^3 + 0.14139875\, x^2 - 0.52953999\, x + 1.72595635$$

$$PN_3(x) = 1.127625965 + (0.093107186)\cdot(x-2) + (0.044399802)\cdot(x-2)\cdot(x-2.5) + (-0.014183696)\cdot(x-2)\cdot(x-2.5)\cdot(x-3)$$

$$H_3(x) = 1.127625965 + (0.093107186)\cdot(x-2) + (0.044399802)\cdot(x-2)^2 + (-0.014183696)\cdot(x-2)^2\cdot(x-2.5)$$

The three interpolation methods are ocillatory and this shows in our above model. Another option we can explore is the Cubic Spline interpolation method. This technique looks to connect the points (in our case connecting x_0 `&x1 `=spline1,x_1& x_2 = spline 2 and x_2& x_3 = spline 3) through cubic models in a way that the models generated for each spline are differentiable at the points where they connect as well. Since we now know the underlying model we will derive the derivative for the endpoints to the cubic spline model.

By using this technique the model for the first spline is:

$$\cosh\left(\frac{1}{2}\right) - 0.130273826\, y + 0.260547652 + 0.09300768246\, (y-2)^{4.962651195}$$

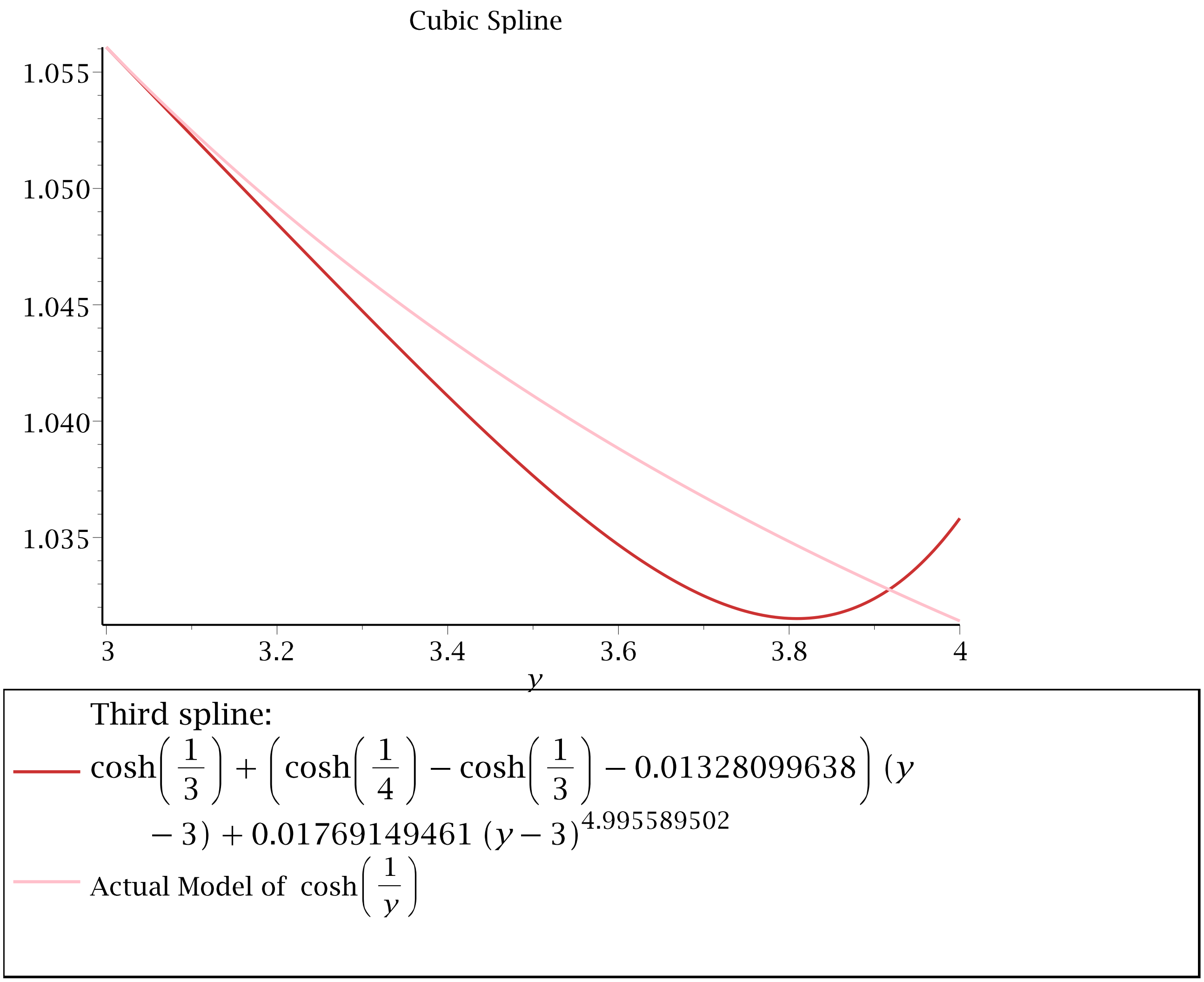
The model for the second spline is:

$$1.081072372 + \left(2.000000000 \cosh\left(\frac{1}{3}\right) - 2.177421485\right) (y-2.5) + 0.03698447484\, (y-2.5)^{4.987138013}$$

The model for the third spline is:

$$\cosh\left(\frac{1}{3}\right) + \left(\cosh\left(\frac{1}{4}\right) - \cosh\left(\frac{1}{3}\right) - 0.01328099638\right) (y-3) + 0.01769149461\, (y-3)^{4.995589502}$$

Since the point we wanted to evaluate was 3.5 and it is is between the points x_2& x_3 we will use the third spline to evaluate our point. When we do we get the value of 1.037656538. This gives us an absolute error of .003438207 and a relative error of 0.003302491. In our example it turns out that the Lagrange interpolating polynomials had the least absolute and relative error with the spline leading close behind.



In our example we started with only four data points. We added the derivates of the endpoints to get our cubic spline model which was useful as well. However, in real life we should strive to get as much data as possible. More data points within the range you are analyzing would prove helpful. If you are able to obtain the derivatives at the data points as well, this will enhance your ability to analyze the model and produce a more accurate model.

References

Faires, J. D., & Burden, R. L. (1998). Numerical methods. Pacific Grove, CA: Brooks/Cole Pub. Co.