How can "The Intermediate Value Theorem" help to locate an interval containing a solution to a nonlinear equation?

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The Intermediate Value Theorem can help to locate intervals containing a solution to a nonlinear equation. Before we explore this further I wanted to take a step back and first define what a continuous function is:
"A function $f: A \to \mathbb{R}$ is continuous at a point $c \in A$, if for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|x - c| < \delta$ $(\& \, x \in A)$ it follows that $|f(x) - f(c)| < \epsilon$." (Abbott, S., Pg 109 (2000))
So we can see by the above definition that if a function is continuous in = then there are no gaps or holes at any point in the function. This is important because the Intermediate Value Theorem makes specific reference to this. The definition of the Intermediate Value Theorem is:
"If $f: [a, b] \to \mathbb{R}$ is continuous, and if L is a real number satisfying $f(a) < L < f(b)$ or $f(a) > L > f(b)$, then there exists a point $c \in (a, b)$, where $f(c) = L$. " (Abbott, S., Pg 120 (2000))

So now that we know exactly what the Intermediate Value Theorem is how can we use this to locate an interval containing a solution to a nonlinear equation? One approach is to see if one can find a linear equation that closely approximates the nonlinear function in the specified interval.(Greenberg, M. D. (2012)) (Perhaps using Taylor Series?) If we know this we can look for the minimum and maximum points of this linear function by deriving its' first derivative and finding where it is equal to zero. (Assuming it is differentiable) If we used the Taylor Series then we will have a polynomial and we will know that it is differentiable everywhere. (Anton, H., Bivens, I., & Davis, S. (2002))
One simple example of this is with the non-linear equation for a pendulum with is
' $\frac{d^2\theta}{dt^2} + \left(\frac{g}{l}\right)\sin(\theta) = 0$" (Greenberg, M. D. Pg 179, (2012)).

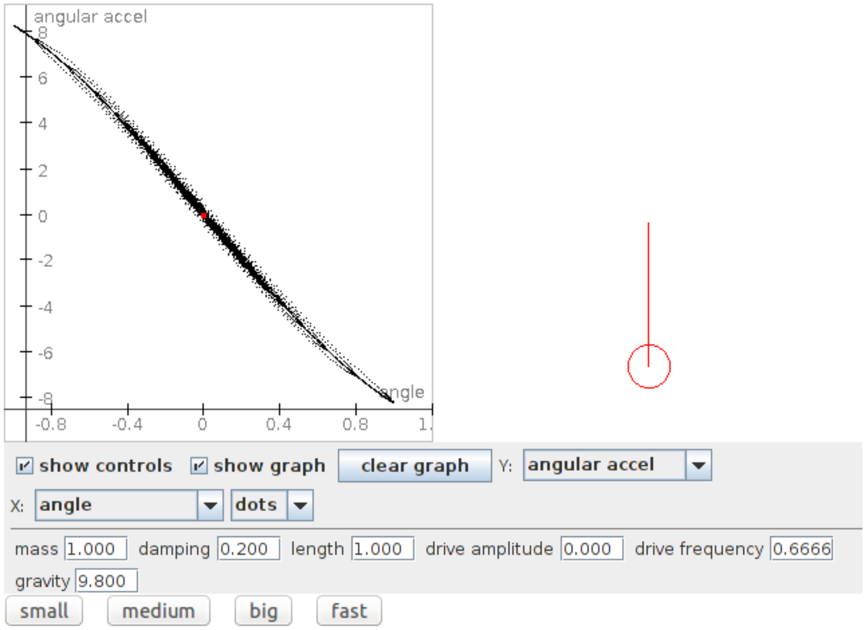
Here g represents gravity and l is the length. We will state that $\varrho = \left(\frac{g}{l}\right)$ and will set the constant ϱ here to $\varrho = (0.2)$. If we consider the MacLauren Series for $\sin(x)$ we get " $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$"(Anton, H., Bivens, I., & Davis, S. Pg 708, (2002)) If we wanted a very local approximation we can just use x and our expression gets converted to:

$\frac{d^2\theta}{dt^2} + (\varrho)\cdot\theta = 0$ or $\frac{d^2\theta}{dt^2} + (0.2)\cdot\theta = 0$

I used the applet from http://www.myphysicslab.com/pendulum1.html to run a simulation of a pendulum. As we can see it is nonlinear but if we take the linear approximation of this we can get the maximum and minimum points for the model.

MyPhysicsLab – Simple Pendulum

This simulation shows a simple pendulum operating under gravity. For small oscillations the pendulum is linear, but it is non-
You can change parameters in the simulation such as mass, gravity, and friction (damping). You can drag the pendulum wit
instructions for enabling Java. Scroll down to see the math!

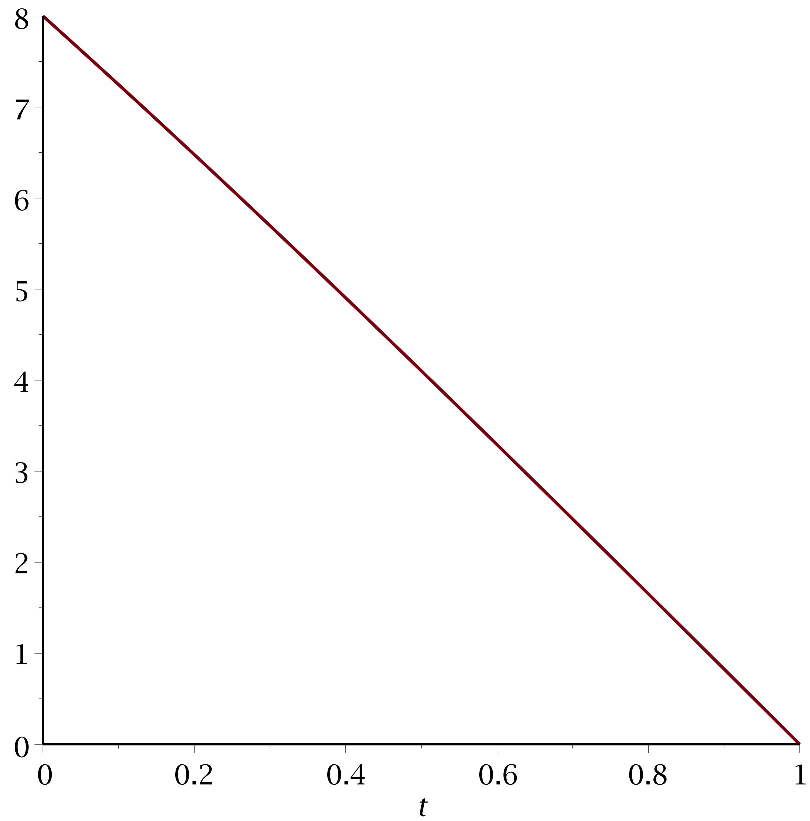


We can use our general linearized model and specify based on the simulation above what we want y to be at time zero and at time 1 to generate the linear model.

$GenLinearApprox := dsolve(\{diff(y(t), t\$2) + (0.2) \cdot y(t) = 0, y(0) = 8, y(1) = 0\}, y(t));$

$$y(t) = -\frac{8\cos\left(\frac{1}{5}\sqrt{5}\right)\sin\left(\frac{1}{5}\sqrt{5}\,t\right)}{\sin\left(\frac{1}{5}\sqrt{5}\right)} + 8\cos\left(\frac{1}{5}\sqrt{5}\,t\right) \quad (1)$$

$plot\left(-\frac{8\cos\left(\frac{1}{5}\sqrt{5}\right)\sin\left(\frac{1}{5}\sqrt{5}\,t\right)}{\sin\left(\frac{1}{5}\sqrt{5}\right)} + 8\cos\left(\frac{1}{5}\sqrt{5}\,t\right), t = 0..1\right);$



$$LinModel := \left(- \frac{8 \cos\left(\frac{1}{5}\sqrt{5}\right) \sin\left(\frac{1}{5}\sqrt{5}\, t\right)}{\sin\left(\frac{1}{5}\sqrt{5}\right)} + 8 \cos\left(\frac{1}{5}\sqrt{5}\, t\right) \right):$$

$$Max := maximize(\mathrm{abs}(LinModel),\ t = 0\,..1);$$

$$8 \tag{2}$$

$$Min := minimize(\mathrm{abs}(LinModel),\ t = 0\,..1);$$

$$0 \tag{3}$$

So here is our approximation we have the maximum of 8 and the minimum of 0 and we have used the generalized linearized model of the oscillator to approximate and locate the interval for the nonlinear oscillator model.

References
Abbott, S. (2000). Understanding Analysis. Springer.
Anton, H., Bivens, I., & Davis, S. (2002). Calculus. New York [u.a.: Wiley.

Greenberg, M. D. (2012). Ordinary differential equations. Hoboken, N.J: Wiley.
Pendulum Physics Simulation. (n.d.). Retrieved from http://www.myphysicslab.com/pendulum1.html