Proof That A Group Of Order 85 Will Be Abelian

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We assume G is a group where the order of $|G| = 85$. We will provide a direct proof that G has a normal Sylow subgroup for some prime p dividing its order. First, we consider the prime factorization for 85 which is (5)(17). Sylows theorem states that there exists a $Syl_p(G)$ subgroup for each prime dividing the order of the group G. By applying this theorem to each of the primes we get:

$$n_5 | (17) \ \ n_5 \equiv 1(\mathbf{mod}\, 5) \Rightarrow n_5 = 1$$

$$n_{17} | (5) \ \ n_{17} \equiv 1(\mathbf{mod}\, 17) \Rightarrow n_{17} = 1$$

By just applying Sylows theorem we see that $n_5 = 1 \,\&\, n_{17} = 1$ which implies $Syl_5(G) \trianglelefteq G \,\&\, Syl_{17}(G) \trianglelefteq G$. Therefore we conclude that group of order 85 has a normal Sylow $p-subgroup$ for primes $5 \,\&\, 17$ dividing its order.

By proposition 13 section 3.2 (Dummit, D. S., & Foote, R. M.,pg 93, (2004)) we know that $\left|Syl_5(G)Syl_{17}(G)\right| = \frac{(|Syl_5(G)|\cdot|Syl_{17}(G)|)}{(|Syl_5(G)\cap Syl_{17}(G)|)}$. Since per Lagrange's theorem $(|Syl_5(G)\cap Syl_{17}(G)|)$ must divide $(|Syl_5(G)|\cdot|Syl_{17}(G)|)$ and we know the only divisor is 1 then $(|Syl_5(G)\cap Syl_{17}(G)|) = 1$. We can now see that the total number of elements in our group is $\left(\frac{(5\cdot 17)}{1}\right) = 85$.

We can now consider $\frac{G}{C_G(P_5)} \cong H \leq 4$ so by Lagrange's Theorem $H \Big| 4 \,\&\, \left(\frac{G}{C_G(P_5)}\right) \Big| 4$. Since we 85 elments in G we must conclude that$C_G(P_5) = |G| \,\&\, \frac{G}{C_G(P_5)} = 1$ .So we now know that all the elements in commute with $P_5 \,\&\, P_5 \leq Z(G)$.

We can now consider $\frac{G}{C_G(P_{17})} \cong K \leq 16$ so by Lagrange's Theorem $K \Big| 16 \,\&\, \left(\frac{G}{C_G(P_5)}\right) \Big| 16$. Since we 85 elments in G we must conclude that$C_G(P_{17}) = |G|$ $\&\ \frac{G}{C_G(P_{17})} = 1$ .So we now know that all the elements in commute with $P_{17} \,\&\, P_{17} \leq Z(G)$.

Therefore, $\frac{G}{Z(G)}$ is cyclic with generator $5\, Z(G)$, and every element can be written in the form $5^1 \cdot 7$. Also, $\frac{G}{Z(G)}$ is cyclic with generator 7 $Z(G)$, and every element can be written in the form $7^1 \cdot 5$.

Therefore, since $\frac{G}{Z(G)}$ is cyclic group G of order 85, $G \cong Z_{85}$ and is abelian.

References

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