Proof that $\mathrm{D}_{16}, Z_2 \, x \, \mathrm{D}_8, Z_2 \, x \, Q_8, QD_{16} \,\&\, M$ Are Isomorphic Non-Abelian Groups Of Order 16

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We will prove that $\mathrm{D}_{16}, Z_2 \, x \, \mathrm{D}_8, Z_2 \, x \, Q_8, QD_{16} \,\&\, M$ are isomorphic non-abelian groups of order 16.

We know that in D_{16} there is a subgroup $\langle r \rangle$ of order 8, in $Z_2 \, x \, \mathrm{D}_8$ there is a subgroup D_8 of the order 8, in $Z_2 \, x \, Q_8$ there is a subgroup Q_8 of the order 8, in QD_{16} there is a subgroup $\langle \sigma \rangle$ of order 8, and in M there is a subgroup $\langle v \rangle$ of order 8.

We then know through Sylow's theorem that these groups are in the order $p^{\alpha} \cdot m = 2^1 \cdot (2^3)$.

The prime 2 is of the order p^1 and by applying Sylow's theorem so that with $\alpha = 1 \,\&\, m = (8)$ we then can see that

$$n_2 | (8) \;\; n_2 \equiv 1 (\mathbf{mod}\, 2) \Rightarrow n_2 = 1$$

So each of these groups has a $Syl_2(Subgroup)$ that I will call T such that $T \trianglelefteq \mathrm{D}_{16}, Z_2 \, x \, \mathrm{D}_8, Z_2 \, x \, Q_8, QD_{16} \,\&\, M.$

We can let E represent the related subgroups of order eight $\left(\langle r \rangle, \mathrm{D}_8, Q_8, \langle \sigma \rangle, \langle v \rangle, \right)$.

We can now consider proposition 8 in section 5.4 "Let H and K be subgroups of the group G. The number of distince ways of wriing each element of the set HK in the form hk, for some $h \in H \,\&\, k \in K$ is $|H \cap K|$. In particular if $|H \cap K| = 1$, then each element of HK can be written uniquely as a product of hk, for some $h \in H$ and $k \in K$." (Dummit, D. S., & Foote, R. M.,pg 171 (2004))

Using proposition 8, for every $t \in T \,\&\, e \in E$ we can write TE as a unique set such that the map $te \mapsto (t, e)$ is a set bijection from TE to $T \rtimes E$.

Therefore, we conclude that $\mathrm{D}_{16}, Z_2 \, x \, \mathrm{D}_8, Z_2 \, x \, Q_8, QD_{16} \,\&\, M$ are isomorphic non-abelian groups of order 16.

References

Dummit, D. S., & Foote, R. M. (2004). Abstract algebra. Hoboken, NJ: Wiley.
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